Reduced Order Finite Time Observers and Output Feedback for Time-Varying Nonlinear Systems*

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Abstract

We provide finite time reduced order observers for a class of nonlinear time-varying continuous-time systems. We use the observers to design globally asymptotically stabilizing output feedback controls. We illustrate our work in a tracking dynamics for a nonholonomic system in chained form.

Key words: Observer, stabilization, finite time, time-varying

1 Introduction

This work continues our search for ways to estimate solutions of systems. This is an important problem, because solving it can make it possible to design output feedback stabilizing controls. The Luenberger observer from Luenberger (1979) is one of many observers for nonlinear systems. However, most existing observers usually only ensure asymptotic convergence of the estimation error to 0, and this can be an obstacle to their implementation.

By definition, a finite time observer is one that provides an exact value of the state that is being estimated after a finite time. This finite time may depend on the initial state (as in Du *et al.* (2013); Perruquetti *et al.* (2008)), or it may be a fixed time that could be independent of the initial state as in Lopez-Ramirez *et al.* (2018). Other finite time observers use past output values or a dynamic extension. This later type of observers was proposed for linear systems, e.g., in Engel and Kreisselmeier (2002), Menold *et al.* (2003), and Raff and Allgower (2008). See also the finite time observers in Mazenc *et al.* (2015) and Sauvage *et al.* (2007) for nonlinear systems.

This paper is motivated by the fact that time-varying systems frequently arise, e.g., by recasting tracking problems as problems for time-varying systems whose goal is to uniformly globally asymptotically stabilize a zero equilibrium, and because measured state components need not be estimated. Here, we adapt Mazenc *et al.* (2015) and Sauvage *et al.* (2007) to build finite time reduced order observers for a class of nonlinear time-varying systems. As in (Bonnans and Rouchon, 2005, Chapt. 4, Sec. 4.4.3) and Friedland (2009), our observers only estimate unmeasured variables. This can produce simpler or better performing observers, and is helpful because when one needs formulas for fundamental solutions of time-varying systems, it is advantageous to consider smaller dimensions.

We believe that our work is the first to provide finite time reduced order observers. Another advantage of this work is that our main observer provides fixed time convergence that is independent of the initial state. It improves on our conference version Mazenc *et al.* (2018*b*) by adding sufficient conditions for our assumptions, a design based on dynamic extensions that yields a formula for the estimation of the state without distributed terms, an output feedback stabilization theorem, and a nonholonomic example that applies our output stabilization theorem, which were not included in Mazenc *et al.* (2018*b*).

We use the following standard notation. The dimensions of our Euclidean spaces are arbitrary, unless otherwise noted. The usual Euclidean norm and the induced matrix norm are denoted by $|\cdot|$, $|\cdot|_\infty$ is the sup norm, $|\cdot|_J$ is the sup over a set J, and I is the identity matrix. We use the standard comparison function classes $\mathcal{KL}$ and $\mathcal{K}_\infty$ and input-to-

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state stable (or ISS), properness, and positive definiteness definitions; see (Khalil, 2002, Chapter 4) and Malisoff and Mazenc (2009). A function $g : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m$ is called locally Lipschitz in the second variable uniformly in the first variable provided there is a function $\alpha \in \mathcal{K}_\infty$ such that for all constants $R > 0$, we have $|g(t, x) - g(t, y)| \leq \alpha(R)|x - y|$ for all $t \in \mathbb{R}$, $x \in \mathcal{B}(R)$, and $y \in \mathcal{B}(R)$, where $\mathcal{B}(R)$ is the closed ball of radius R centered at 0 in the usual Euclidean norm. A function $V : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ is called uniformly proper and positive definite provided there exist functions $\underline{\alpha} \in \mathcal{K}_\infty$ and $\overline{\alpha} \in \mathcal{K}_\infty$ such that $\underline{\alpha}(|x|) \leq V(t, x) \leq \overline{\alpha}(|x|)$ for all $t \geq 0$ and $x \in \mathbb{R}^n$. We assume for simplicity that the initial times for our solutions are always $t_0 = 0$, unless otherwise noted. For any piecewise continuous function $\Omega : \mathbb{R} \to \mathbb{R}^{n \times n}$, let Φ_Ω be the unique function such that the following conditions hold for all $t \in \mathbb{R}$ and $t_0 \in \mathbb{R}$:

$$\frac{\partial \Phi_\Omega}{\partial t}(t, t_0) = -\Phi_\Omega(t, t_0)\Omega(t) \text{ and } \Phi_\Omega(t_0, t_0) = I.$$

Then $\Phi_\Omega^{-1}(t, s) = \Phi_\Omega(s, t)$ holds for all real s and t, and $\mathcal{M}_\Omega(t, s) = \Phi_\Omega^{-1}(t, s)$ is the fundamental solution for Ω and the system $\dot{x} = \Omega(t)x$; see (Sontag, 1998, Lemma C.4.1). We also use the following generalization of (Mazenc *et al.*, 2018*b*, Lemma 2) which we prove in the appendix:

Lemma 1 *Let $\mathcal{A} \in \mathbb{R}^{n \times n}$ be a constant matrix and let $\mathcal{E} : \mathbb{R} \to \mathbb{R}^{n \times n}$ be a bounded piecewise continuous function. Let $\mathcal{M}_{\mathcal{A}+\mathcal{E}}$ denote the fundamental solution of*

$$\dot{\zeta}(t) = [\mathcal{A} + \mathcal{E}(t)]\, \zeta(t). \tag{1}$$

Then for all $t \in \mathbb{R}$ and $s \in \mathbb{R}$, the inequalities

$$\left|\mathcal{M}_{\mathcal{A}+\mathcal{E}}(t, s) - e^{\mathcal{A}(t-s)}\right| \leq |\mathcal{E}|_\infty |t - s| e^{(|\mathcal{A}|+|\mathcal{E}|_\infty)|t-s|} \tag{2}$$

and $|\mathcal{M}_{\mathcal{A}+\mathcal{E}}(t, s)| \leq e^{|t-s|(|\mathcal{A}|+|\mathcal{E}|_\infty)}$ are satisfied. □

2 Main Observer Design for Time-Varying Systems

2.1 Statement of Result and Remarks

We study nonlinear systems with outputs of the form

$$\begin{cases} \dot{z}(t) = A_1(t)x_r(t) + \delta_1(t, z(t)) \\ \dot{x}_r(t) = A_2(t)x_r(t) + \delta_2(t, z(t)) \\ y(t) = z(t) \end{cases} \tag{3}$$

where z is valued in $\mathbb{R}^p$, x_r is valued in $\mathbb{R}^{n-p}$, A_i for $i = 1$ and 2 is piecewise continuous and bounded, and our conditions on δ_1 and δ_2 will be specified below; see Remark 1 for the motivation for (3). We assume:

Assumption 1 *There exist a constant $\tau > 0$ and a bounded matrix valued function $L : \mathbb{R} \to \mathbb{R}^{(n-p)\times p}$ of class C^1 with a bounded first derivative such that with the choice $H(t) = A_2(t) + L(t)A_1(t)$, the following are true: (i) The matrix*

$$\kappa(t) = \Phi_H(t, t - \tau) - \Phi_{A_2}(t, t - \tau) \tag{4}$$

is invertible for all $t \in \mathbb{R}$ and (ii) the inverse function $\kappa^{-1}(t)$ is a bounded function of t. □

Assumption 2 *The δ_i's are piecewise continuous with respect to t and locally Lipschitz with respect to z. The system (3) is forward complete.* □

See Section 2.3 below on ways to check Assumption 1. We introduce the function

$$\delta_\sharp(t, z) = L(t)\delta_1(t, z) + \delta_2(t, z) + \dot{L}(t)z - H(t)L(t)z \tag{5}$$

where H and L are from Assumption 1, and the dynamic extensions

$$\begin{cases} \dot{\gamma}_1(t) = H(t)\gamma_1(t) + \delta_\sharp(t, z(t)) \\ \dot{\gamma}_2(t) = A_2(t)\gamma_2(t) + \delta_2(t, z(t)), \end{cases} \tag{6}$$

which are reminiscent of the ones used in Mazenc *et al.* (2015). In terms of the observer

$$\begin{aligned} x_r^*(t) = {} & \kappa(t)^{-1} [L(t - \tau)z(t - \tau) \\ & -\Phi_H(t, t - \tau)L(t)z(t) \\ & +\Phi_H(t, t - \tau)\gamma_1(t) - \gamma_1(t - \tau)] \\ & -\kappa(t)^{-1}[\Phi_{A_2}(t, t - \tau)\gamma_2(t) - \gamma_2(t - \tau)] \end{aligned} \tag{7}$$

for all $t \geq \tau$, we prove the following, but see Remark 2 on the implementability of the observer, and see Remark 3 for generalizations that allow external disturbances and measurement noise (but where instead of a finite time observer, we get an observation error depending on sup norms of the disturbances and of the measurement noise):

Theorem 1 *Let L, A_1, A_2 and τ be such that (3) satisfies Assumptions 1-2. Then*

$$x_r(t) = x_r^*(t) \tag{8}$$

holds for all solutions of (3)-(6) for all $t \geq \tau$ and all initial conditions. If, in addition, the functions A_1, A_2 and L are periodic of period $T > 0$ and $\tau = T$, then $\kappa(T) = \Phi_H(T, 0) - \Phi_{A_2}(T, 0)$ and

$$\begin{aligned} x_r^*(t) = {} & \kappa(T)^{-1} [L(t)z(t - T) - \Phi_H(T, 0)L(t)z(t) \\ & +\Phi_H(T, 0)\gamma_1(t) - \gamma_1(t - T)] \\ & -\kappa(T)^{-1}[\Phi_{A_2}(T, 0)\gamma_2(t) - \gamma_2(t - T)] \end{aligned} \tag{9}$$

holds for all $t \geq T$ and all constant initial functions $\gamma(0) \in \mathbb{R}^{2(n-p)}$ and $(z(0), x_r(0)) \in \mathbb{R}^n$.

Remark 1 *To motivate (3), consider the class of nonlinear systems $\dot{x}(t) = Ax(t) + \delta(t, y(t))$ where A is a constant matrix and δ is uniformly locally Lipschitz in y uniformly in t, with an output $y(t) = Cx(t)$ that is valued in $\mathbb{R}^p$ with $p < n$ where C is of full rank and where the pair (A, C) is observable. Since C has full rank, (Luenberger, 1979, pp. 304-306) (with $\delta(t, y)$ added to the right side) proves that there are constant matrices C_T and A_1 and A_2, a linear change of coordinates $x_T = C_T x = [y^\top, x_r^\top]^\top$ and functions δ_i that are uniformly locally Lipschitz in y uniformly in t such that the x_T system can be written as the special case*

$$\begin{cases} \dot{y}(t) = A_1 x_r(t) + \delta_1(t, y(t)) \\ \dot{x}_r(t) = A_2 x_r(t) + \delta_2(t, y(t)) \end{cases} \tag{10}$$

of (3) with (A_2, A_1) observable. Since (A_2, A_1) is observable, (Mazenc et al.*, 2015, Lemma 1) provides an L and a $\tau > 0$ so that $\kappa = e^{-A_2\tau} - e^{-H\tau}$ with $H = A_2 + LA_1$ is invertible; this is done by picking L so that all eigenvalues of H are negative, real, and smaller than the real parts of the eigenvalues of $-A_2$, and then picking τ large enough so that $|e^{\tau H}||e^{-\tau A_2}| < 1$. Hence, Assumption 1 holds for (10).*

In fact, we can allow arbitrarily small constants $\tau > 0$, by the following approach. First, choose a matrix L and a constant $\tau_0 > 0$ such that $\kappa = e^{-A_2\tau_0} - e^{-H\tau_0}$ with $H = A_2 + LA_1$ is invertible, i.e., such that $\mathcal{D}(\tau) = \mathrm{determinant}(e^{-A_2\tau} - e^{-H\tau})$ is nonzero at $\tau = \tau_0$. Then, for our fixed L and any constant $\bar{\tau} \in (0, \tau_0)$, we can find a constant $\tau_* \in (0, \bar{\tau})$ such that $\mathcal{D}(\tau_*) \neq 0$, so Assumption 1 holds with this τ_*. The existence of τ_* follows from the real analyticity of $\mathcal{D}$, because if there were a $\bar{\tau} \in (0, \tau_0)$ such that no such $\tau_* \in (0, \bar{\tau})$ existed, then $\mathcal{D}(\tau) = 0$ for all $\tau \in (0, \bar{\tau})$, and then an analytic continuation argument would give the contradiction $\mathcal{D}(\tau_0) = 0$. Hence, we can eliminate the requirement that $\tau > 0$ is large enough. □

Remark 2 *The observer (7) can be computed in practice from the known y measurements and the known δ_i's when Φ_H and Φ_{A_2} are available. Besides, the advantages of the formula (9) are important. First, there is no integral term in it (which is due to the use of the dynamic extension (6)). Second, in the periodic case that is described in Theorem 1, the constant matrices $\kappa(T)^{-1}$, $\Phi_H(T, 0)$ and $\Phi_{A_2}(T, 0)$ can be determined through software. In fact, since*

$$\mathcal{M}_H(T, 0) = [\phi_H(T, 0, e_1)...\phi_H(T, 0, e_{n-p})]$$

where the ith column $\phi_H(T, 0, e_i)$ is the solution of the initial value problem $\dot{Z} = H(t)Z$, $Z(0) = e_i$, for all i evaluated at T, where $e_i \in \mathbb{R}^{n-p}$ is the ith standard basis vector (by the linearity of the system $\dot{Z} = H(t)Z$), we can compute $\mathcal{M}_H(T, 0)$ (and so also its inverse $\Phi_H(T, 0)$) by solving $n-p$ initial value problems. The same applies to $\mathcal{M}_{A_2}(T, 0)$. □

Remark 3 *Our proof of Theorem 1 in Section 2.2 below is easily generalized to dynamics with external perturbations and measurement noise, as follows. If we add uncertainties $f_1(t)$, $f_2(t)$, and $\epsilon(t)$ to $\dot{z}(t)$, $\dot{x}_r(t)$, and $y(t)$ respectively in (3), where the f_i's and ϵ piecewise continuous and locally bounded, and if we replace the local Lipschitzness condition in Assumption 2 by global Lipschitzness with respect to z, and if we replace the z values in (6)-(7) by the corresponding output values $y(t) = z(t) + \epsilon(t)$ with the measurement noise ϵ, then similar arguments to the ones in Section 2.2 (using the second conclusion of Lemma 1 and the boundedness of H and A_2 to get $\sup_{t\geq 0}\sup_{\ell\in[t-\tau,t]} |\mathcal{M}_H(t-\tau, \ell)| < \infty$ and $\sup_{t\geq 0}\sup_{\ell\in[t-\tau,t]} |\mathcal{M}_{A_2}(t-\tau, \ell)| < \infty$) provide a function $\gamma_e \in \mathcal{K}_\infty$ such that $|x_r^*(t) - x_r(t)| \leq \gamma_e(|(f_1, f_2, \epsilon)|_{[0,t]})$ holds for all $t \geq \tau$ and all initial conditions.* □

2.2 Proof of Theorem 1

Assumption 2 ensures that the system (3) is forward complete. We deduce that the solutions are defined for all $t \geq 0$. Next, let us introduce

$$s(t) = x_r(t) + L(t)z(t). \tag{11}$$

Simple calculations give

$$\begin{aligned}\dot{s}(t) &= A_2(t)x_r(t) + \delta_2(t, z(t)) \\ &\quad + \dot{L}(t)z(t) + L(t)\dot{z}(t) \\ &= H(t)x_r(t) + L(t)\delta_1(t, z(t)) + \delta_2(t, z(t)) \\ &\quad + \dot{L}(t)z(t) \\ &= H(t)s(t) + \delta_\sharp(t, z(t)),\end{aligned} \tag{12}$$

where $\delta_\sharp$ is defined in (5). By applying variation of parameters to

$$\begin{cases} \dot{s}(t) = H(t)s(t) + \delta_\sharp(t, z(t)) \\ \dot{x}_r(t) = A_2(t)x_r(t) + \delta_2(t, z(t)) \end{cases} \tag{13}$$

we obtain

$$\begin{cases} \Phi_H(t, t-\tau)s(t) = s(t-\tau) \\ + \int_{t-\tau}^t \mathcal{M}_H(t-\tau, \ell)\delta_\sharp(\ell, z(\ell))\mathrm{d}\ell \ \text{ and} \\ \Phi_{A_2}(t, t-\tau)x_r(t) = x_r(t-\tau) \\ + \int_{t-\tau}^t \mathcal{M}_{A_2}(t-\tau, \ell)\delta_2(\ell, z(\ell))\mathrm{d}\ell. \end{cases} \tag{14}$$

By subtracting the equalities in (14), we obtain

$$\begin{aligned}&\Phi_H(t, t-\tau)[x_r(t) + L(t)z(t)] - \Phi_{A_2}(t, t-\tau)x_r(t) \\ &= x_r(t-\tau) + L(t-\tau)z(t-\tau) \\ &+ \int_{t-\tau}^t \mathcal{M}_H(t-\tau, \ell)\delta_\sharp(\ell, z(\ell))\mathrm{d}\ell - x_r(t-\tau) \\ &- \int_{t-\tau}^t \mathcal{M}_{A_2}(t-\tau, \ell)\delta_2(\ell, z(\ell))\mathrm{d}\ell\end{aligned} \tag{15}$$

which gives

$$\begin{aligned}\kappa(t)x_r(t) &= L(t-\tau)z(t-\tau) - \Phi_H(t, t-\tau)L(t)z(t) \\ &\quad + \int_{t-\tau}^t \mathcal{M}_H(t-\tau, \ell)\delta_\sharp(\ell, z(\ell))\mathrm{d}\ell \\ &\quad - \int_{t-\tau}^t \mathcal{M}_{A_2}(t-\tau, \ell)\delta_2(\ell, z(\ell))\mathrm{d}\ell.\end{aligned} \tag{16}$$

By applying variation of parameters to (6), we obtain

$$\begin{aligned}&\int_{t-\tau}^t \mathcal{M}_H(t-\tau, \ell)\delta_\sharp(\ell, z(\ell))\mathrm{d}\ell \\ &= \Phi_H(t, t-\tau)\gamma_1(t) - \gamma_1(t-\tau) \ \text{ and} \\ &\int_{t-\tau}^t \mathcal{M}_{A_2}(t-\tau, \ell)\delta_2(\ell, z(\ell))\mathrm{d}\ell \\ &= \Phi_{A_2}(t, t-\tau)\gamma_2(t) - \gamma_2(t-\tau).\end{aligned} \tag{17}$$

It follows that

$$\begin{aligned}\kappa(t)x_r(t) &= L(t-\tau)z(t-\tau) - \Phi_H(t, t-\tau)L(t)z(t) \\ &\quad + \Phi_H(t, t-\tau)\gamma_1(t) - \gamma_1(t-\tau) \\ &\quad - \Phi_{A_2}(t, t-\tau)\gamma_2(t) + \gamma_2(t-\tau).\end{aligned} \tag{18}$$

Consequently (8) is satisfied. In the particular case where the functions A_1, A_2 and L are periodic of period $T = \tau$ then for all $t \in \mathbb{R}$, $\kappa(t)^{-1} = \kappa(T)^{-1}$, $\Phi_H(t, t-\tau) = \Phi_H(T, 0)$, $L(t-\tau) = L(t)$ and $\Phi_{A_2}(t, t-\tau) = \Phi_{A_2}(T, 0)$. This allows us to conclude.

2.3 Checking Assumption 1

In several cases, one can verify Assumption 1.

1) If $n-p=1$, then we can apply variation of parameters to get Φ_{A_2} and Φ_H in explicit forms.

2) Let us assume that the functions A_1, A_2, and L are periodic of period $T=\tau$ and that

$$\kappa(T)=\Phi_H(T,0)-\Phi_{A_2}(T,0) \tag{19}$$

is invertible. Then $\kappa(t)=\kappa(\tau)$ is invertible for all $t\in\mathbb{R}$ so Assumption 1 is satisfied with $\tau=T$. The invertibility can be checked in practice by computing $\Phi_H(T,0)$ and $\Phi_{A_2}(T,0)$ as explained in Remark 2.

3) Next, let us assume that there are an observable pair $(A_{02},A_{01})\in\mathbb{R}^{(n-p)\times(n-p)}\times\mathbb{R}^{p\times(n-p)}$ of constant matrices and functions Δ_i such that $A_i(t)=A_{0i}+\Delta_i(t)$ for $i=1,2$. Then one can determine a matrix L_0 and a constant $\overline{\delta}>0$ such that if $|\Delta_i|_\infty\leq\overline{\delta}$, $i=1,2$, then Assumption 1 is satisfied with $L(t)=L_0$. Indeed, in this case one can use (Mazenc *et al.*, 2015, Lemma 1) to find a constant matrix L_0 such that

$$\kappa_0=e^{-(A_{02}+L_0A_{01})\tau}-e^{-A_{02}\tau} \tag{20}$$

is invertible. By writing $\kappa(t)$ as

$$\begin{aligned}\kappa(t)&=\kappa_0+\left[\Phi_H(t,t-\tau)-e^{-(A_{02}+L_0A_{01})\tau}\right]\\&\quad-\left[\Phi_{A_2}(t,t-\tau)-e^{-A_{02}\tau}\right]\\&=\kappa_0\left[I+R(t)\right]\end{aligned} \tag{21}$$

with

$$\begin{aligned}R(t)&=\kappa_0^{-1}\left[\Phi_H(t,t-\tau)-e^{-(A_{02}+L_0A_{01})\tau}\right]\\&\quad-\kappa_0^{-1}\left[\Phi_{A_2}(t,t-\tau)-e^{-A_{02}\tau}\right],\end{aligned} \tag{22}$$

we can use Lemma 1 to prove that

$$|R|_\infty\leq\overline{c}(\overline{\delta})\overline{\delta},\ \text{where} \tag{23}$$

$$\begin{aligned}&\overline{c}(\overline{\delta})=\\&|\kappa_0^{-1}|\left[e^{(|A_{02}|+\overline{\delta})\tau}+(1+|L_0|)e^{\left(|H_0|+(1+|L_0|)\overline{\delta}\right)\tau}\right]\tau\end{aligned} \tag{24}$$

and $H_0=A_{02}+L_0A_{01}$. Thus $|\kappa|_\infty\leq|\kappa_0|(1+\overline{\delta}\overline{c}(\overline{\delta}))$. If, in addition, $\overline{\delta}<1/\overline{c}(\overline{\delta})$, then we can check that $I+R(t)$ is invertible for all $t\in\mathbb{R}$ (by checking that its null space is trivial). Since κ_0 is invertible, it follows that $\kappa(t)$ is invertible for all $t\in\mathbb{R}$. Then

$$\kappa^{-1}(t)=(I+R(t))^{-1}\kappa_0^{-1}. \tag{25}$$

Since

$$(I+R(t))^{-1}=\sum_{k=0}^{\infty}(-1)^kR(t)^k \tag{26}$$

we deduce that

$$\left|(I+R(t))^{-1}\right|\leq\sum_{k=0}^{+\infty}(\overline{c}(\overline{\delta})\overline{\delta})^k\leq\frac{1}{1-\overline{c}(\overline{\delta})\overline{\delta}}. \tag{27}$$

Hence, $|\kappa^{-1}|_\infty\leq\frac{|\kappa_0^{-1}|}{1-\overline{c}(\overline{\delta})\overline{\delta}}$, so Assumption 1 is satisfied.

3 Output Feedback Stabilization

In this section, we use the observer from the previous section to solve a dynamic output feedback stabilization problem.

3.1 Assumptions and Statement of Main Result.

We study

$$\left\{\begin{aligned}\dot z(t)&=A_1(t)x_r(t)+B_1(t)u(t)\\&\quad+\rho_1(t,z(t))+f_1(t)\\\dot x_r(t)&=A_2(t)x_r(t)+B_2(t)u(t)\\&\quad+\rho_2(t,z(t))+f_2(t)\end{aligned}\right. \tag{28}$$

where z is valued in $\mathbb{R}^p$, x_r is valued in $\mathbb{R}^{n-p}$, the output is $y(t)=z(t)$, A_i and B_i for $i=1,2$ are known piecewise continuous bounded matrix valued functions, $\rho=(\rho_1,\rho_2)$ is known and piecewise continuous with respect to t, and $f=(f_1,f_2)$ is an unknown locally bounded piecewise continuous function. We assume:

Assumption 3 *There exist a function $u_s(t,\chi)$ that is locally Lipschitz in $\chi=(z,x_r)$ uniformly in t, a C^1 uniformly proper positive definite function V, positive constants c_1 and c_2, and $\gamma\in\mathcal{K}_\infty$ so that for all choices of the locally bounded piecewise continuous functions $\mu=(\mu_1,\mu_2)$ and $h=(h_1,h_2)$ and all $t\geq0$, the following hold: (1) The time derivative of V along all solutions of*

$$\left\{\begin{aligned}\dot z(t)&=A_1(t)x_r(t)+B_1(t)u(t)\\&\quad+\rho_1(t,z(t))+h_1(t)\\\dot x_r(t)&=A_2(t)x_r(t)+B_2(t)u(t)\\&\quad+\rho_2(t,z(t))+h_2(t)\end{aligned}\right. \tag{29}$$

in closed loop with the state feedback $u(t)=u_s(t,x_r(t)+\mu_1(t),z(t)+\mu_2(t))$ satisfies

$$\dot V(t)\leq-c_1V(t,\chi(t))+\gamma(|(\mu,h)(t)|) \tag{30}$$

and (2) its time derivative along all trajectories χ of (29) in closed loop with $u(t)=0$ satisfies

$$\dot V(t)\leq c_2V(t,\chi(t))+\gamma(|h(t)|) \tag{31}$$

for all $t\geq0$. □

Assumption 4 *The function $\rho=(\rho_1,\rho_2)$ is locally Lipschitz in its second variable uniformly in t and there is a function $\alpha\in\mathcal{K}_\infty$ such that $|\rho(t,a)|\leq\alpha(|a|)$ for all $a\in\mathbb{R}^p$ and $t\geq0$.* □

The preceding assumptions are satisfied if the ρ_i's have the linear forms $\rho_i(t,z)=\rho_{i,\star}(t)z$ with continuous bounded functions $\rho_{i,\star}(t)$ for $i=1,2$ and if in addition the system $\dot\chi=Q_1(t)\chi+Q_2(t)u$ with the choices $\chi=(z,x_r)$,

$$Q_1=\begin{bmatrix}\rho_{1*}&A_1\\\rho_{2*}&A_2\end{bmatrix}, \tag{32}$$

and $Q_2=[B_1^\top\ B_2^\top]^\top$ admits a bounded piecewise continuous function K_Q such that $\dot{\chi} = (Q_1(t) + Q_2(t)K_Q(t))\chi$ is uniformly globally exponentially stable to 0. This is done by using the quadratic Lyapunov function for this closed-loop system provided by (Khalil, 2002, Theorem 4.14) and $u_s(t,\chi) = K_Q(t)\chi$.

Setting

$$\rho_4(t,z) = -[D(t)z + \rho_3(t,z)], \tag{33}$$

where

$$\rho_3(t,z) = L(t)\rho_1(t,z) + \rho_2(t,z) \tag{34}$$

and

$$D(t) = \dot{L}(t) - H(t)L(t), \tag{35}$$

and with H, L, and κ from Assumption 1, we prove this ISS result:

Theorem 2 *Let τ, L, H, u_s, κ, c_1, and c_2 be such that Assumptions 1, 3, and 4 hold. Then we can construct $\bar{\beta} \in \mathcal{KL}$ and $\bar{\gamma} \in \mathcal{K}_\infty$ such that: All solutions $\chi(t)$ of (28), in closed loop with $u(t) = u_\star(t, \overline{x}_r(t), y(t))$ where*

$$u_\star(t, \overline{x}_r(t), y(t)) = \begin{cases} u_s(t, \overline{x}_r(t), y(t)) & \text{when } t \geq \tau \\ 0 & \text{when } t < \tau \end{cases} \tag{36}$$

and where $\overline{x}_r$ is

$$\begin{aligned} \overline{x}_r(t) &= \kappa(t)^{-1}\left[L(t-\tau)z(t-\tau)\right. \\ &\quad -\Phi_H(t,t-\tau)L(t)z(t) + \Phi_H(t,t-\tau)\omega_1(t) \\ &\quad \left.-\omega_1(t-\tau)\right] \\ &\quad -\kappa(t)^{-1}[\Phi_{A_2}(t,t-\tau)\omega_2(t) - \omega_2(t-\tau)] \\ \dot{\omega}_1(t) &= H(t)\omega_1(t) \\ &\quad + [L(t)B_1(t) + B_2(t)]\, u_\star(t, \overline{x}_r(t), y(t)) \\ &\quad +\rho_3(t, z(t)) + D(t)z(t) \\ \dot{\omega}_2(t) &= A_2(t)\omega_2(t) + B_2(t)u_\star(t, \overline{x}_r(t), y(t)) \\ &\quad +\rho_2(t, z(t)) \end{aligned} \tag{37}$$

are such that

$$|\chi(t)| \leq \bar{\beta}(|\chi(0)|, t) + \bar{\gamma}(|f|_{[0,t]}) \tag{38}$$

holds for all $t \geq 0$ and all constant initial functions $\omega(0) \in \mathbb{R}^{2(n-p)}$, $\overline{x}_r(0) \in \mathbb{R}^{n-p}$, and $(z(0), x_r(0)) \in \mathbb{R}^n$. □

3.2 Proof of Theorem 2

Let us consider the system (29) in closed-loop with (36). First, let us observe that (31) ensures that for any solution of this system, there is $s > \tau$ such that the solution is defined over $[0, s)$. Now, let

$$\begin{aligned} \delta_1(t) &= B_1(t)u_\star(t, \overline{x}_r(t), y(t)) + \rho_1(t, z(t)) + f_1(t), \\ \delta_2(t) &= B_2(t)u_\star(t, \overline{x}_r(t), y(t)) + \rho_2(t, z(t)) + f_2(t) \end{aligned} \tag{39}$$

and

$$\begin{cases} \dot{\gamma}_1(t) = H(t)\gamma_1(t) + L(t)\delta_1(t, z(t)) + \delta_2(t) \\ \qquad\qquad +D(t)z(t), \\ \dot{\gamma}_2(t) = A_2(t)\gamma_2(t) + \delta_2(t). \end{cases} \tag{40}$$

Then arguing as we did to prove Theorem 1, we deduce that for all $t \in [\tau, s)$,

$$\begin{aligned} &x_r(t) = \kappa(t)^{-1}\left[L(t-\tau)z(t-\tau)\right. \\ &\left.-\Phi_H(t,t-\tau)L(t)z(t) + \Phi_H(t,t-\tau)\gamma_1(t) - \gamma_1(t-\tau)\right] \\ &-\kappa(t)^{-1}[\Phi_{A_2}(t,t-\tau)\gamma_2(t) - \gamma_2(t-\tau)]. \end{aligned} \tag{41}$$

Now, we observe that $\varrho_i = \gamma_i - \omega_i$ for $i = 1, 2$ satisfy

$$\begin{cases} \dot{\varrho}_1(t) = H(t)\varrho_1(t) + f_3(t) \ \text{ and} \\ \dot{\varrho}_2(t) = A_2(t)\varrho_2(t) + f_2(t), \end{cases} \tag{42}$$

where $f_3(t) = L(t)f_1(t) + f_2(t)$. By applying variation of parameters, we obtain

$$\begin{aligned} &\Phi_H(t,t-\tau)\varrho_1(t) - \varrho_1(t-\tau) \\ &= \textstyle\int_{t-\tau}^t \mathcal{M}_H(t-\tau,\ell)f_3(\ell)\mathrm{d}\ell \ \text{ and} \\ &\Phi_{A_2}(t,t-\tau)\varrho_2(t) - \varrho_2(t-\tau) \\ &= \textstyle\int_{t-\tau}^t \mathcal{M}_{A_2}(t-\tau,\ell)f_2(\ell)\mathrm{d}\ell \end{aligned} \tag{43}$$

for all $t \in [\tau, s)$. Thus,

$$\begin{aligned} &\Phi_H(t,t-\tau)\gamma_1(t) - \gamma_1(t-\tau) = \Phi_H(t,t-\tau)\omega_1(t) \\ &-\omega_1(t-\tau) + \textstyle\int_{t-\tau}^t \mathcal{M}_H(t-\tau,\ell)f_3(\ell)\mathrm{d}\ell, \\ &\Phi_{A_2}(t,t-\tau)\gamma_2(t) - \gamma_2(t-\tau) = \Phi_{A_2}(t,t-\tau)\omega_2(t) \\ &-\omega_2(t-\tau) + \textstyle\int_{t-\tau}^t \mathcal{M}_{A_2}(t-\tau,\ell)f_2(\ell)\mathrm{d}\ell. \end{aligned} \tag{44}$$

Combining (41) and (44), we obtain

$$\begin{aligned} x_r(t) &= \kappa(t)^{-1}\left[L(t-\tau)z(t-\tau)\right. \\ &\quad -\Phi_H(t,t-\tau)L(t)z(t) + \Phi_H(t,t-\tau)\omega_1(t) \\ &\quad \left.-\omega_1(t-\tau)\right] \\ &\quad -\kappa(t)^{-1}[\Phi_{A_2}(t,t-\tau)\omega_2(t) - \omega_2(t-\tau)] \\ &\quad +\kappa(t)^{-1}\textstyle\int_{t-\tau}^t \mathcal{M}_H(t-\tau,\ell)f_3(\ell)\mathrm{d}\ell \\ &\quad -\kappa(t)^{-1}\textstyle\int_{t-\tau}^t \mathcal{M}_{A_2}(t-\tau,\ell)f_2(\ell)\mathrm{d}\ell. \end{aligned} \tag{45}$$

From (37), it follows that

$$x_r(t) = \overline{x}_r(t) + \varsigma(t) \tag{46}$$

with

$$\begin{aligned} \varsigma(t) &= \kappa(t)^{-1}\textstyle\int_{t-\tau}^t \mathcal{M}_H(t-\tau,\ell)[L(\ell)f_1(\ell) + f_2(\ell)]\mathrm{d}\ell \\ &\quad -\kappa(t)^{-1}\textstyle\int_{t-\tau}^t \mathcal{M}_{A_2}(t-\tau,\ell)f_2(\ell)\mathrm{d}\ell. \end{aligned} \tag{47}$$

It follows that for all $t \in [\tau, s)$, the closed-loop system is

$$\begin{aligned} \dot{z}(t) &= A_1(t)x_r(t) + B_1(t)u_\star(t, x_r(t) - \varsigma(t), y(t)) \\ &\quad +\rho_1(t, z(t)) + f_1(t) \\ \dot{x}_r(t) &= A_2(t)x_r(t) + B_2(t)u_\star(t, x_r(t) - \varsigma(t), y(t)) \\ &\quad +\rho_2(t, z(t)) + f_2(t). \end{aligned} \tag{48}$$

Now, from Assumption 3, it follows that

$$\dot{V}(t) \leq -c_1 V(t, \chi(t)) + \gamma(|(-\varsigma(t), 0, f_1(t), f_2(t))|) \tag{49}$$

for all $t \in [\tau, s)$ and

$$\dot{V}(t) \leq c_2 V(t, \chi(t)) + \gamma(|f(t)|) \tag{50}$$

for all $t \in [0, \tau]$. Since V is uniformly proper positive definite, we deduce that $s = +\infty$ and for all $t \in [0, \tau]$,

$$\begin{aligned} V(t, \chi(t)) \;&\leq\; e^{c_2\tau} V(0, \chi(0)) + e^{c_2\tau} \textstyle\int_0^t \gamma(|f(\ell)|)\mathrm{d}\ell \\ &\leq\; e^{c_2\tau} V(0, \chi(0)) + e^{c_2\tau} \tau\gamma(|f|_{[0,t]}) \end{aligned} \tag{51}$$

and for all $t > \tau$,

$$\begin{aligned} V(t, \chi(t)) &\leq e^{-c_1(t-\tau)} V(\tau, \chi(\tau)) \\ &\quad + \textstyle\int_\tau^t e^{c_1(\ell-t)} \gamma(|(-\varsigma(\ell), 0, f_1(\ell), f_2(\ell))|)\mathrm{d}\ell \\ &\leq e^{-c_1(t-\tau)} V(\tau, \chi(\tau)) \\ &\quad + \textstyle\int_\tau^t e^{c_1(\ell-t)} \gamma(2|\varsigma(\ell)|)\mathrm{d}\ell \\ &\quad + \textstyle\int_\tau^t e^{c_1(\ell-t)} \gamma(2|(f_1(\ell), f_2(\ell))|)\mathrm{d}\ell \\ &= e^{-c_1(t-\tau)} V(\tau, \chi(\tau)) \\ &\quad + \textstyle\int_\tau^t e^{c_1(\ell-t)} \gamma(2|\varsigma(\ell)|)\mathrm{d}\ell \\ &\quad + \textstyle\int_\tau^t e^{c_1(\ell-t)} \mathrm{d}\ell \gamma(2|f|_{[0,t]}), \end{aligned} \tag{52}$$

by the bound $\gamma(a+b) \leq \gamma(2a) + \gamma(2b)$ for suitable a and b.

From the formula of ς in (47) and Lemma 1, we deduce that

$$\begin{aligned} |\varsigma(t)| \;&\leq\; |\kappa^{-1}|_\infty \Big(e^{\tau|H|_\infty} \textstyle\int_{t-\tau}^t [|L|_\infty |f_1(\ell)| + |f_2(\ell)|]\mathrm{d}\ell \\ &\quad + e^{\tau|A_2|_\infty} \textstyle\int_{t-\tau}^t |f_2(\ell)|\mathrm{d}\ell \Big) \\ &\leq\; |\kappa^{-1}|_\infty \tau \left[e^{\tau|H|_\infty} |L|_\infty \sup_{s\in[t-\tau,t]} |f_1(s)| \right. \\ &\quad \left. + \left(e^{\tau|H|_\infty} + e^{\tau|A_2|_\infty}\right) \sup_{s\in[t-\tau,t]} |f_2(s)| \right] \\ &\leq\; \bar{b} \sup_{s\in[t-\tau,t]} |f(s)|, \end{aligned} \tag{53}$$

where

$$\bar{b} = |\kappa^{-1}|_\infty \left[e^{\tau|H|_\infty} |L|_\infty + e^{\tau|H|_\infty} + e^{\tau|A_2|_\infty} \right] \tau.$$

Then for all $t > \tau$,

$$\begin{aligned} V(t, \chi(t)) \;&\leq\; e^{-c_1(t-\tau)} V(\tau, \chi(\tau)) \\ &\quad + \textstyle\int_\tau^t e^{c_1(\ell-t)} \gamma\left(2\bar{b}|f|_{[0,t]}\right) \mathrm{d}\ell \\ &\quad + \textstyle\int_\tau^t e^{c_1(\ell-t)} \mathrm{d}\ell \gamma(2|f|_{[0,t]}) \\ &\leq\; e^{-c_1(t-\tau)} V(\tau, \chi(\tau)) + \tfrac{1}{c_1} \gamma\left(2\bar{b}|f|_{[0,t]}\right) \\ &\quad + \tfrac{1}{c_1} \gamma(2|f|_{[0,t]}). \end{aligned} \tag{54}$$

This inequality and (51) yield

$$\begin{aligned} V(t, \chi(t)) \;&\leq\; e^{-c_1(t-\tau)} \left[e^{c_2\tau} V(0, \chi(0)) \right. \\ &\quad \left. + e^{c_2\tau} \tau\gamma(|f|_{[0,t]}) \right] + \tfrac{1}{c_1} \gamma\left(2\bar{b}|f|_{[0,t]}\right) \\ &\quad + \tfrac{1}{c_1} \gamma(2|f|_{[0,t]}) \\ &\leq\; e^{-c_1 t + (c_1+c_2)\tau} V(0, \chi(0)) + \gamma_\dagger(|f|_{[0,t]}) \end{aligned} \tag{55}$$

for all $t \geq \tau$ with

$$\gamma_\dagger(m) = e^{c_2\tau} \tau\gamma(m) + \frac{1}{c_1} \gamma\left(2\bar{b}m\right) + \frac{1}{c_1} \gamma(2m). \tag{56}$$

Moreover from the second inequality of (51), we deduce that, for all $t \in [0, \tau]$,

$$\begin{aligned} V(t, \chi(t)) \;&\leq\; e^{-c_1 t + (c_1+c_2)\tau} V(0, \chi(0)) \\ &\quad + e^{c_2\tau} \tau\gamma(|f|_{[0,t]}). \end{aligned} \tag{57}$$

It follows that

$$V(t, \chi(t)) \;\leq\; e^{-c_1 t + (c_1+c_2)\tau} V(0, \chi(0)) + \gamma_\dagger(|f|_{[0,t]}) \tag{58}$$

for all $t \geq 0$. The properties of V ensure that there are two functions $\mathcal{P}_i$, $i = 1, 2$ of class $\mathcal{K}_\infty$ such that

$$\mathcal{P}_1(|\chi|) \leq V(t, \chi) \leq \mathcal{P}_2(|\chi|) \tag{59}$$

for all $t \in \mathbb{R}$ and $\chi \in \mathbb{R}^n$. These inequalities and (58) yield

$$\begin{aligned} |\chi(t)| \;&\leq\; \mathcal{P}_1^{-1}\left(e^{-c_1 t + (c_1+c_2)\tau} \mathcal{P}_2(|\chi(0)|) + \gamma_\dagger(|f|_{[0,t]})\right) \\ &\leq\; \mathcal{P}_1^{-1}\left(2e^{-c_1 t + (c_1+c_2)\tau} \mathcal{P}_2(|\chi(0)|)\right) \\ &\quad + \mathcal{P}_1^{-1}\left(2\gamma_\dagger(|f|_{[0,t]})\right) \end{aligned} \tag{60}$$

for all $t \geq 0$. Since the function $\gamma_\dagger$ is of class $\mathcal{K}_\infty$, we can conclude.

4 Application to Nonholonomic System in Chained Form

4.1 Tracking Problem

We illustrate Theorem 2 using this variant of a system from (Malisoff and Mazenc, 2009, p. 143):

$$\dot{\xi}_4 = \xi_3 v_1, \;\; \dot{\xi}_3 = \xi_2 v_1, \;\; \dot{\xi}_2 = v_2, \;\; \dot{\xi}_1 = v_1 \tag{61}$$

with $(\xi_1, \xi_2, \xi_3, \xi_4)$ valued in $\mathbb{R}^4$ and the input (v_1, v_2) valued in $\mathbb{R}^2$, which is a nonholonomic system in chained form, and where we will omit time arguments t of functions to make the notation more concise. We assume that ξ_4, ξ_3 and ξ_1 are measured, but that ξ_2 is not measured. We design a dynamic output feedback making (61) track the trajectory $\left(\xi_{1r}(t), \xi_{2r}(t), \xi_{3r}(t), \xi_{4r}(t)\right) = \left(t + \frac{1}{2}\sin(t), 0, 0, 0\right)$. We use the change of variables and feedback and $x_1 = \xi_1 - \xi_{1r}(t)$ and $v_1(t, x_1) = -x_1 + 1 + \frac{1}{2}\cos(t)$. This produces the x_1 subsystem $\dot{x}_1 = -x_1$ and so prompts us to solve the problem of globally asymptotically stabilizing the tracking dynamics

$$\begin{aligned} &\dot{\xi}_4 = \left(1 + \tfrac{1}{2}\cos(t)\right)\xi_3, \;\; \dot{\xi}_3 = \left(1 + \tfrac{1}{2}\cos(t)\right)\xi_2, \\ &\dot{\xi}_2 = v_2 \end{aligned} \tag{62}$$

to 0, by replacing x_1 by 0 in the $(\xi_2, \xi_3, \xi_4, x_1)$ dynamics. In terms of the notation of Section 3, the system (61) can be written as

$$\begin{cases} \dot{z}_1(t) = \left(1 + \frac{1}{2}\cos(t)\right) z_2(t) \\ \dot{z}_2(t) = \left(1 + \frac{1}{2}\cos(t)\right) x_r(t) \\ \dot{x}_r(t) = u(t), \end{cases} \tag{63}$$

which has the form (28) with the choices

$$A_1(t) = \begin{pmatrix} 0 \\ 1 + \frac{1}{2}\cos(t) \end{pmatrix}, \qquad (64)$$

$$\rho_1(t,z) = \begin{pmatrix} \left(1 + \frac{1}{2}\cos(t)\right) z_2 \\ 0 \end{pmatrix} \qquad (65)$$

$A_2(t) = 0$, $B_1(t) = 0$, $B_2(t) = 1$, $f_1 = 0$, $f_2 = 0$, and $\rho_2(t,z) = 0$. Let us choose $L(t) = [0 \quad 2]$. This gives $H(t) = A_2(t) + L(t)A_1(t) = 2 + \cos(t)$ and the functions

$$\Phi_H(t,s) = e^{-2(t-s)+\sin(s)-\sin(t)} \text{ and } \Phi_{A_2}(t,s) = 1. \qquad (66)$$

Choosing $\tau = 2$, we obtain

$$\kappa(t) = e^{-4+\sin(t-2)-\sin(t)} - 1. \qquad (67)$$

The inequalities

$$|\kappa|_\infty \leq 1 \text{ and } |1/\kappa|_\infty \leq \frac{e^2}{e^2-1} \qquad (68)$$

hold. It follows that Assumption 1 is satisfied.

4.2 Applying Theorem 2 to (63)

One can easily prove that Assumption 3 is satisfied with

$$u_s(t,x_r,z) = \left(1 + \tfrac{1}{2}\cos(t)\right)(-z_1 - 3z_2 - 3x_r), \qquad (69)$$

by using the Hurwitzness of

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{pmatrix} \qquad (70)$$

to obtain a quadratic choice of V. Assumption 4 is satisfied too. It follows that Theorem 2 applies to (63). This theorem gives the following globally asymptotically stabilizing output feedback for (63):

$$u_\star(t,\overline{x}_r(t),z(t)) = \begin{cases} u_s(t,\overline{x}_r(t),z(t)) & \text{when } t \geq 2 \\ 0 & \text{when } t < 2 \end{cases} \qquad (71)$$

with

$$\begin{aligned} \overline{x}_r(t) &= \tfrac{\mathcal{T}(t)}{e^{-4+\sin(t-2)-\sin(t)}-1} + \tfrac{\omega_2(t-2)-\omega_2(t)}{e^{-4+\sin(t-2)-\sin(t)}-1}, \\ \dot{\omega}_1(t) &= (2+\cos(t))\omega_1(t) + u_\star(t,\overline{x}_r(t),z(t)) \\ &\quad -2(2+\cos(t))z_2(t), \\ \dot{\omega}_2(t) &= u_\star(t,\overline{x}_r(t),z(t)) \text{ and} \end{aligned} \qquad (72)$$

$$\begin{aligned} \mathcal{T}(t) &= 2z_2(t-2) - 2e^{-4+\sin(t-2)-\sin(t)}z_2(t) \\ &\quad + e^{-4+\sin(t-2)-\sin(t)}\omega_1(t) - \omega_1(t-2). \end{aligned} \qquad (73)$$

4.3 Simulations

We performed simulations, which show the efficiency of our approach. Fig. 1 shows the simulation of the system (63) with $u(t) = u_\star(t,\overline{x}_r(t),z(t))$ as defined in (71). Since our simulation shows good stabilization, it helps illustrate our general theory, in the special case of the system (61).

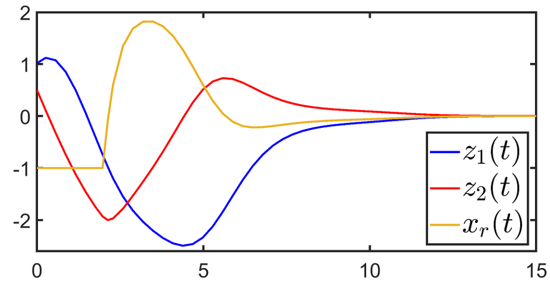

Fig. 1. Simulation of the time varying system (63) with $u(t) = u_\star(t,\overline{x}_r(t),z(t))$. Time unit on horizontal axis is seconds.

5 Conclusions

We designed reduced order finite time dynamic observers and corresponding output feedbacks that are free of distributed control terms. We have exhibited families of systems for which the observer and control law can be easily implemented. We hope to combine Theorem 2 with Mazenc *et al.* (2018*a*) to cover delays and disturbances in the input and intermittent output observations. Extensions pertaining to disturbances on the measurements are expected too.

Appendix: Proof of Lemma 1

For all real values of s and t, the function $z(t,s) = \mathcal{M}_{\mathcal{A}+\mathcal{E}}(t,s) - e^{\mathcal{A}(t-s)}$ satisfies

$$\begin{aligned} \tfrac{\partial}{\partial t}z(t,s) &= (\mathcal{A}+\mathcal{E}(t))\mathcal{M}_{\mathcal{A}+\mathcal{E}}(t,s) - \mathcal{A}e^{\mathcal{A}(t-s)} \\ &= \mathcal{A}z(t,s) + \mathcal{E}(t)\mathcal{M}_{\mathcal{A}+\mathcal{E}}(t,s) \end{aligned} \qquad (74)$$

and $z(s,s) = 0$, so

$$z(t,s) = \int_s^t e^{\mathcal{A}(t-r)}\mathcal{E}(r)\mathcal{M}_{\mathcal{A}+\mathcal{E}}(r,s)\mathrm{d}r, \qquad (75)$$

by a variation of parameters. Also, for all real r and s, the Peano-Baker formula for fundamental matrix solutions (e.g., from (Sontag, 1998, p.489)) gives

$$|\mathcal{M}_{\mathcal{A}+\mathcal{E}}(r,s)| \leq e^{|\mathcal{A}+\mathcal{E}|_\infty|r-s|}, \qquad (76)$$

Set $\underline{s} = \min\{s,t\}$ and $\bar{s} = \max\{s,t\}$. We can combine (76) with (75) to get

$$|z(t,s)| \leq e^{(\bar{s}-\underline{s})|\mathcal{A}|}|\mathcal{E}|_\infty(\bar{s}-\underline{s})e^{|\mathcal{E}|_\infty(\bar{s}-\underline{s})}. \qquad (77)$$

The lemma follows by noting that $\bar{s} - \underline{s} = |t-s|$.

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